

Exponential corrections to low-temperature expansion of $2D$ non-abelian models

O. Borisenko¹, V. Kushnir²

N.N.Bogolyubov Institute for Theoretical Physics, National Academy of Sciences of Ukraine, 252143 Kiev, Ukraine

Abstract

The thermodynamic limit of certain exponential corrections to the weak coupling expansion of two-dimensional models is investigated. The expectation values of operators contributing to the first order coefficient of the low-temperature expansion of the free energy are calculated for the order $O(e^{-\beta})$. They are proven to diverge logarithmically with the volume for non-abelian models.

It has been recently argued that the low-temperature asymptotic expansion of two-dimensional ($2D$) non-abelian models can be non-uniform in the volume [1]. As an example, authors of [1] involved the superinstanton boundary conditions (BC) and have shown that the conventional perturbation theory (PT) produces results for invariant functions which differ in the thermodynamic limit (TL) from those obtained with more standard BC, and the difference starts at $O(1/\beta^2)$. Since the asymptotic expansion of invariant functions in the TL is unique, the discovery of [1] may indicate a failure of the PT. However, it has been shown later that the PT with superinstanton BC diverges for higher orders and it has been argued that the low-temperature asymptotic expansion is non-uniform in the volume only for superinstanton BC while there are good reasons to believe that the PT gives true, uniform asymptotics with conventional BC [2]-[3] (see, however Ref.[4]).

If the low-temperature expansion (which is asymptotic when the volume is fixed) is non-uniform in the volume, one should encounter infrared (IR) divergences (logarithmic in $2D$) either in the coefficients of the expansion or in the remainder to the expansion. It has been proven that the coefficients of the continuum PT done in the dimensional regularization with mass regulator term is IR finite [5]. In this paper we assume the same holds for bare lattice PT with periodic BC. This assumption however cannot influence our result. We are going to show that logarithmic divergences do appear in the exponential corrections to the finite-volume asymptotic expansion, already in the order $O(e^{-\beta})$.

We work with $SU(2)$ model but to make our procedure clearer we perform the same calculations for the XY model where it is known rigorously that the PT gives a

¹email: oleg@ap3.bitp.kiev.ua

²email: kushnir@ap3.bitp.kiev.ua

uniform asymptotic expansion [6]. The arguments are very simple, both conceptually and technically. First, let us discuss the general strategy. In our previous papers [7], [8] we proposed to use an invariant link formulation to perform the low-temperature expansion of $2D$ models. The starting point of calculations is the following partition function of $2D$ $SU(N)$ model on the periodic lattice

$$Z = \int \prod_l dV_l \exp \left[\beta \sum_l \text{Re Tr} V_l + \ln J(V) \right] , \quad (1)$$

where the Jacobian $J(V)$ is a product of $SU(N)$ delta-functions taken over all plaquettes of $2D$ lattice

$$J(V) = \prod_p \left[\sum_r d_r \chi_r \left(\prod_{l \in p} V_l \right) \right] . \quad (2)$$

The sum over r runs over all representations of $SU(N)$, $d_r = \chi_r(I)$ is the dimension of the r -th representation. The $SU(N)$ character χ_r depends on an ordered product of the link matrices V_l along a plaquette

$$\prod_{l \in p} V_l = V_n(x) V_m(x+n) V_n^+(x+m) V_m^+(x) . \quad (3)$$

The similar formulae can be written for the XY model. In either case the formal low-temperature expansion takes the form [7]

$$Z = \left[1 + \sum_{k=1}^{\infty} \frac{1}{\beta^k} A_k(\partial_h, \partial_s) \right] M(h, s) , \quad (4)$$

where A_k are known differential operators acting on the generating functional M . We find up to a constant [7]

$$\begin{aligned} M(h, s) = & \int_{-\infty}^{\infty} \prod_{x,k} d\alpha_k(x) \int_{-\infty}^{\infty} \prod_{l,k} d\omega_k(l) \exp \left[-\frac{1}{2} \omega_k^2(l) - i\omega_k(l) [\alpha_k(x+n) - \alpha_k(x)] \right] \\ & \times \sum_{m(x)=-\infty}^{\infty} \exp \left[2\pi i \lambda \sum_x m(x) \left(\sum_k \alpha_k^2(x) \right)^{1/2} + \sum_{l,k} \omega_k(l) h_k(l) + \sum_{x,k} \alpha_k(x) s_k(x) \right] . \end{aligned} \quad (5)$$

The group index k takes the value $k = 1, 2, 3$ for $SU(2)$ and $k = 1$ for the XY model. The external sources $h_k(l)$ are coupled to the link fields $\omega_k(l)$ while $s_k(x)$ are coupled to the auxiliary fields $\alpha_k(x)$. In the abelian case $s_k(x) = 0$ since the operators A_k depend only on link fields. $\lambda = \sqrt{2\beta}$ for $SU(2)$ and $\lambda = \sqrt{\beta}$ for the XY model.

The finite-volume asymptotic expansion arises from the configuration $\{m_x\} = 0$ for all x (“vacuum” sector). In [8] we have shown that the first two coefficients of the fixed distance correlation function calculated in this vacuum sector coincide with the conventional result. For general reasons (the uniqueness of the asymptotic expansion in the finite volume for a given BC and the IR finiteness of the coefficients of the expansion) one should expect that these coefficients coincide up to an arbitrary order.

All other configurations in the sum over m_x in (5) are exponentially suppressed with β and do not contribute to the asymptotics. We shall however prove that

$$\frac{1}{2L^2}C_{XY}^1 = \frac{1}{2L^2}A_1(\partial_h, \partial_s) M(h, s) = \frac{1}{32} + O(\beta e^{-\pi^2\beta}) , \quad (6)$$

for the XY model (actually, this is true for any A_k) while

$$\frac{1}{2L^2}C_{SU2}^1 = \frac{1}{2L^2}A_1(\partial_h, \partial_s) M(h, s) = \frac{3}{64} + O(\beta \ln L e^{-2\pi^2\beta}) , \quad (7)$$

for $SU(2)$ model. The constants $1/32$ and $3/64$ are the standard first order coefficients of the free energy expansion.

Before we proceed to calculations we would like to present some simple arguments why Eq.(7) should be expected. In the vacuum sector the $SU(N)$ delta-function in Eq.(1) reduces to the Dirac delta-function so that the partition function (1) becomes

$$Z(\beta \gg 1) = \int \prod_l dV_l \exp \left[\beta \sum_l \text{Re Tr} V_l \right] \prod_{p,k} \delta(\omega_p^k) \quad (8)$$

with the zero modes for the auxiliary fields omitted from Green functions [7]. $\omega_k(p)$ is a plaquette angle defined as

$$V_p = \prod_{l \in p} V_l = \exp \left[i \sigma^k \omega_k(p) \right] \quad (9)$$

and has the following expansion

$$\omega_k(p) = \omega_k^{(0)}(p) + \frac{1}{\sqrt{2\beta}} \omega_k^{(1)}(p) + \frac{1}{2\beta} \omega_k^{(2)}(p) + \dots \quad (10)$$

On a dual lattice (see [7]) the delta-function in (8) is expanded as

$$\begin{aligned} \prod_{x,k} \delta(\omega_x^k) &= \int_{-\infty}^{\infty} \prod_{x,k} \frac{d\alpha_k(x)}{2\pi} \exp \left[-i \sum_{x,k} \alpha_k(x) \omega_k^{(0)}(x) \right] \\ &\times \left[1 + \sum_{q=1}^{\infty} \frac{(-i)^q}{q!} \left(\sum_{x,k} \alpha_k(x) \sum_{n=1}^{\infty} \frac{\omega_k^{(n)}(x)}{(2\beta)^{n/2}} \right)^q \right] . \end{aligned} \quad (11)$$

If we adopt the following definitions

$$\omega_k(l) \rightarrow \frac{\partial}{\partial h_k(l)} , \quad \alpha_k(x) \rightarrow \frac{\partial}{\partial s_k(x)} , \quad (12)$$

the generating functional in the vacuum sector takes the form

$$M_0(h, s) = \exp \left[\frac{1}{4} s_k(x) G_{x,x'} s_k(x') + \frac{i}{2} s_k(x) D_l(x) h_k(l) + \frac{1}{4} h_k(l) G_{ll'} h_k(l') \right] , \quad (13)$$

where $G_{ll'}$ and $D_l(x')$ are link Green functions (see the Appendix). These functions are IR finite by construction. It follows from the last equations that the IR finiteness of the first order coefficient requires the finiteness of the expectation value of the following operator

$$U = \frac{1}{2} \left(\sum_{x,k} \alpha_k(x) \omega_k^{(1)}(x) \right)^2, \quad (14)$$

because only this operator includes IR divergent Green function $G_{x,x'}$. One can show that the IR finiteness of this operator is equivalent to

$$\left\langle \left(\sum_{x,k} \omega_k^{(1)}(x) \right)^2 \right\rangle_0 = 0, \quad (15)$$

where $\langle \dots \rangle_0$ refers to the following partition function

$$Z_0 = \int_{-\infty}^{\infty} \prod_{l,k} d\omega_k(l) \exp \left[-\frac{1}{2} \omega_k^2(l) \right] \prod_{x,k} \delta \left(\omega_k^{(0)}(x) \right). \quad (16)$$

An even power of the real operator $\sum_{x,k} \omega_k^{(1)}(x)$ calculated over positive measure can only be zero if the operator itself vanishes identically on all configurations contributing to (16). Indeed, the constraint in (16)

$$\omega_k^{(0)}(x) = 0 \quad (17)$$

can be trivially solved on the original lattice in terms of site variables

$$\begin{aligned} \omega_k(l_3) &= \omega_k(x) - \omega_k(x+n), \quad \omega_k(l_4) = \omega_k(x+n) - \omega_k(x+n+m), \\ \omega_k(l_1) &= \omega_k(x+m) - \omega_k(x+n+m), \quad \omega_k(l_2) = \omega_k(x) - \omega_k(x+m). \end{aligned} \quad (18)$$

For this solution, $\sum_{x,k} \omega_k^{(1)}(x) = 0$. In [8] we have shown how Eq.(15) is fulfilled in terms of link Green functions (see also below).

The crucial point is that the IR finiteness of the first order coefficient requires the finiteness of the operator U not only in the vacuum sector but for all m_x in Eq.(5). For an arbitrary configuration $\{m_x\}$ the constraint (17) becomes

$$\left(\sum_k (\omega_k^{(0)}(x))^2 \right)^{1/2} = 2\pi \lambda m_x. \quad (19)$$

It is easy to find a non-trivial solution of (19) on which the operator $\sum_{x,k} \omega_k^{(1)}(x)$ does not vanish (e.g., the “vacuum” solution (18) for the components $k = 1, 2$ and some non-trivial vortex solution for $\omega_3(l)$). Since the integration measure is positive for any configuration $\{m_x\}$, there is no way for Eq.(15) to be satisfied in “non-vacuum” sectors. This argument explains why Eq.(7) is not surprising but should be expected on general grounds.

We want to prove now Eq.(6). Calculating the generating functional (5) for the XY model we find (sum over all repeating indices is understood)

$$M_{XY}(h) = e^{\frac{1}{4}h_l G_{ll'} h_{l'}} \sum_{m_x} \delta(\sum_x m_x) e^{-\pi^2 \beta m_x G_{x,x'} m_{x'} + \pi \sqrt{\beta} h_l D_l(x') m_{x'}} . \quad (20)$$

Actually, Eq.(6) follows directly from this representation for $M_{XY}(h)$: the generating functional depends only on functions which are IR finite. The operator A_1 is given by $\frac{1}{24} \sum_l (\partial^4 / \partial h_l^4)$. It leads to the following expansion

$$\begin{aligned} \frac{1}{2L^2} A_1 (\partial_h) M_{XY}(h) &= Z_{vor} \left[\frac{1}{32} - \frac{\pi^2 \beta}{16L^2} \sum_{x_1 x_2} D(x_1 - x_2) \langle m_{x_1} m_{x_2} \rangle_{vor} \right. \\ &\quad \left. + \frac{\pi^4 \beta^2}{48L^2} \sum_{x_1 x_2 x_3 x_4} \sum_l D_l(x_1) D_l(x_2) D_l(x_3) D_l(x_4) \langle m_{x_1} m_{x_2} m_{x_3} m_{x_4} \rangle_{vor} \right] . \end{aligned} \quad (21)$$

Here, the expectation value $\langle \dots \rangle_{vor}$ refers to the vortex part of the XY model

$$Z_{vor} = \sum_{m_x} \delta(\sum_x m_x) e^{-\pi^2 \beta m_x G_{x,x'} m_{x'}} . \quad (22)$$

The first exponential correction comes from the configuration $m_x = \delta_{xy} - \delta_{xz}$. We find

$$\frac{1}{L^2} \sum_{x_1 x_2} D(x_1 - x_2) \langle m_{x_1} m_{x_2} \rangle_{vor} = \frac{2}{L^2} \sum_{y \neq z} D(y - z) e^{-2\pi^2 \beta D(y-z)} \left[1 + O(e^{-\beta}) \right] , \quad (23)$$

and similar expression can be written for the last term in (21). Since $D(y - z) \geq 1/2$ for $y \neq z$, Eq.(6) follows.

Let us now turn to the $SU(2)$ model. It seems to be a difficult task to calculate the generating functional for an arbitrary configuration $\{m_x\}$ because of the square root in (5). We therefore restrict ourself to the first exponential correction which again arises from $m_x = \delta_{xy} - \delta_{xz}$, i.e. we write

$$\sum_{m(x)=-\infty}^{\infty} \exp[2\pi i \lambda \sum_x m(x) \alpha(x)] = 1 + 4 \sum_{y \neq z} \cos(2\pi \lambda \alpha(y)) \cos(2\pi \lambda \alpha(z)) + \dots .$$

Using the inverse Laplace transform

$$\cos(2a^{1/2} y^{1/2}) = \frac{1}{2i} \left(\frac{y}{\pi} \right)^{1/2} \int_{C-i\infty}^{C+i\infty} \frac{dt}{\sqrt{t}} e^{yt-a/t} , \quad C > 0 ,$$

the generating functional can be reduced to the form

$$M_{SU(2)}(h, s) = M_0(h, s) \left(1 + 4 \sum_{y \neq z} M_1(h, s) + O(e^{-4\pi^2 \beta}) \right) , \quad (24)$$

where M_0 is given in (13). An integral form of M_1 reads ($G_0 \equiv G_{x,x}$)

$$M_1(h, s) = -\frac{\pi\beta}{2G_{yz}} e^{-4\pi^2\beta G_0} \int_{C-i\infty}^{C+i\infty} \frac{dt ds}{(ts-1)^{3/2}} (tG_{yz} - G_0)(sG_{yz} - G_0) \exp \left[2\pi^2\beta G_{yz}(t+s) + \frac{1}{4G_{yz}(ts-1)} (ta_k^2(z) + sa_k^2(y) - 2a_k(z)a_k(y)) \right]. \quad (25)$$

We use the notation ($h_k(l) \equiv h_k(n; x)$)

$$a_k(y) = \sum_x G_{xy} [h_k(x) - is_k(x)] , \quad h_k(x) = \sum_{n=1}^2 [h_k(n; x) - h_k(n; x-n)] .$$

As an example, let us give an expression for M_1 at vanishing sources

$$M_1(0, 0) = \frac{1}{2} e^{-4\pi^2\beta D(y-z)} \left(1 - \frac{4\pi^2\beta}{G_{yz}} D^2(y-z) \right) + O(e^{-8\pi^2\beta G_0}) . \quad (26)$$

One sees that the convergence to the TL is very slow, like $O(1/\ln L)$.

The proof of Eq.(7) is now straightforward. The operator A_1 reads [7]

$$A_1 = \frac{1}{24} \sum_l \left(\sum_k \omega_k^2(l) \right)^2 - \frac{1}{3} \sum_l \left(\sum_k \omega_k^2(l) \right) - i \sum_x \sum_k \alpha_k(x) \omega_k^{(2)}(x) - U , \quad (27)$$

where U is given in (14). In [7] we have shown that

$$\frac{1}{2L^2} A_1 (\partial_h, \partial_s) M_0(h, s) = \frac{3}{64} . \quad (28)$$

It is seen from (27) that the IR divergent function $G_{x,x'}$ can arise only from the third and fourth terms. It is easy to prove that the operator $\sum_x \sum_k \alpha_k(x) \omega_k^{(2)}(x)$ is IR finite. Therefore, it is sufficient to study the contribution from the last term in (27). We find

$$\frac{1}{2L^2} U (\partial_h, \partial_s) M_{SU(2)}(h, s) = U_{div} + U_{fin} . \quad (29)$$

U_{fin} denotes a finite contribution which depends only on IR finite link Green functions. For U_{div} we find (up to the terms vanishing in the TL)

$$U_{div} = \frac{1}{2L^2} \sum_{x,x'} G_{x,x'} \left[\frac{3}{16} W_{x,x'}^{(0)} + 5\pi^2\beta \sum_{yz} e^{-4\pi^2\beta D(y-z)} W_{x,x'}^{(1)}(y, z) \right] , \quad (30)$$

where

$$W_{x,x'}^{(0)} = \sum_{i < j}^4 \sum_{i' < j'}^4 (G_{l_i, l'_{i'}} G_{l_j, l'_{j'}} - G_{l_i, l'_{j'}} G_{l_j, l'_{i'}}) , \quad (31)$$

$$W_{x,x'}^{(1)}(y, z) = \sum_{i < j}^4 \sum_{i' < j'}^4 (G_{l_i, l'_{i'}} B_{l_j, l'_{j'}} - G_{l_i, l'_{j'}} B_{l_j, l'_{i'}} + G_{l_j, l'_{j'}} B_{l_i, l'_{i'}} - G_{l_j, l'_{i'}} B_{l_i, l'_{j'}}) , \quad (32)$$

$$B_{l,l'}(y, z) = (D_l(y) - D_l(z)) (D_{l'}(y) - D_{l'}(z)) . \quad (33)$$

The link l_i ($l'_{j'}$) refers to one of four links attached to a given site x (x'). Let us substitute $G_{x,x'} = G_0 - D(x - x')$ in (30). Since the function $D(x - x')$ is IR finite we conclude that the IR finiteness of Eq.(30) requires

$$\sum_{x,x'} W_{x,x'}^{(0)} = 0 , \quad \sum_{x,x'} W_{x,x'}^{(1)}(y, z) = 0 . \quad (34)$$

The first of these equations is nothing but another form of Eq.(15). Its validity can be proven directly. Using (46) we rewrite (34) as

$$\sum_{x,x'} W_{x,x'}^{(0)} = \frac{1}{4} \sum_{b_1 b_2} A(b_1 b_2) [A(b_1 b_2) - A(b_2 b_1)] , \quad (35)$$

where sums over b_i run over all links of the lattice and

$$A(b_1 b_2) = \sum_x \sum_{i < j} G_{b_1 l_i} G_{b_2 l_j} .$$

The equality $A(b_1 b_2) - A(b_2 b_1) = 0$ follows from (43) and (45). This equality ensures the IR finiteness of the first order coefficient in the vacuum sector.

However, the second equation in (34) cannot be satisfied, in accordance with our general arguments. Indeed, using the same tricks as above we get

$$\sum_{x,x'} W_{x,x'}^{(1)}(y, z) = \frac{1}{2} \sum_b (F_b(y) - F_b(z))^2 , \quad (36)$$

where

$$F_b(y) = \sum_x \sum_{i < j} (G_{b l_i} D_{l_j}(y) - G_{b l_j} D_{l_i}(y)) . \quad (37)$$

It follows from the last two equations that in order to ensure the IR finiteness, the function $F_b(y)$ must be y -independent. In fact, $F_b(y)$ does depend on y as it follows from the definition of functions G_{bl} and $D_l(y)$. To see this dependence directly, one can use Eqs.(43), (44) (see the Appendix) for the link Green functions and perform calculations in momentum space. From (36) it also follows that $\sum_{x,x'} W_{x,x'}^{(1)}(y, z)$ depends only on a difference $(y - z)$. Since $W_{x,x'}^{(1)}(y, y) = 0$ we end up with (omitting finite terms)

$$\frac{1}{2L^2} \sum_{x,x'} G_{x,x'} \sum_{yz} e^{-4\pi^2 \beta D(y-z)} W_{x,x'}^{(1)}(y, z) = G_0 e^{-2\pi^2 \beta} \sum_{x,x'} W_{x,x'}^{(1)}(0, 1) + O(e^{-c\pi^2 \beta}), \quad (38)$$

where $c > 2$. Since $G_0 \approx \frac{1}{\pi} \ln L$, this completes the proof of Eq.(7).

We have also computed the second term in (30) numerically. For $\beta = 2$ we find the values $3.036 \cdot 10^{-15}$ for $L = 200$, $3.267 \cdot 10^{-15}$ for $L = 300$ and $3.558 \cdot 10^{-15}$ for $L = 500$. These numbers are so small compared to the first order coefficient $3/128$ (at $\beta = 2$) that even if there are deviations from the conventional PT in the TL, it seems there is no chance to see them in the Monte-Carlo simulations.

Let us add some comments.

I. $2D$ $SU(N)$ principal chiral models are essentially compact models whose Gibbs measure obeys a certain periodicity required by the invariant group measure. As a rule, the weak coupling expansion needs a construction of the Gaussian ensemble which destroys the compactness and periodicity of the original measure. When the volume is fixed the corrections coming from the extension of the integration region to infinity and from the breaking of periodicity are exponentially small and thus do not contribute to the asymptotic expansion. The subtle question is what happens with these corrections in the large volume limit, in particular whether they are infrared finite or not. In our opinion, the link formulation of these models presents a good tool for resolving this problem. Summation over m_x in the expression for the generating functional (5) appears due to using group invariant delta-function. In the abelian case it corresponds to the summation over vortex configurations. Probably, some similar interpretation can be given in the $SU(2)$ case (it is, of course, also tempting to speculate that these configurations are responsible for a mass gap generation similar to vortices in the XY model). In this paper we have investigated certain exponential corrections to the first order coefficient in the expansion of the free energy which appear from the mentioned summation. Our main result, Eq.(7), shows that the IR divergences do not cancel for non-abelian models.

However since such divergences appear only in exponentially suppressed terms one could ask how significant they are, in particular for the construction of the continuum limit. One considers asymptotics obtained by letting L go to infinity as a certain function of β , e.g. for the XY model one takes $L = O(\exp \sqrt{\beta})$. The proof of the asymptoticity then follows from certain inequalities and the fact that the PT coefficients with Dirichlet and free BC differ only by exponentially small corrections for sufficiently large β (see section 4 of [6]). A strategy of a similar proof for non-abelian models was outlined in [3]. Since one can take a power dependence on β for L , the second term on the right-hand side of Eq.(7) remains properly bounded. If however one is interested in the limit $L \rightarrow \infty$ where L is β -independent, we conclude that the coefficients of the low-temperature expansion are not bounded uniformly in L .

II. The second type of exponential corrections comes from the restriction of the integration region over link variables. For example, the one-link integral in the XY model is given by

$$\begin{aligned} I_l &= \int_{-\pi\sqrt{\beta}}^{\pi\sqrt{\beta}} d\omega(l) \exp \left[-\frac{1}{2}\omega^2(l) - i\omega(l)[\alpha(x+n) - \alpha(x)] \right] \\ &= (2\pi)^{1/2} e^{-\frac{1}{2}(\alpha(x+n) - \alpha(x))^2} \left(\operatorname{erf} \left(\pi\sqrt{\beta/2} + \frac{i}{\sqrt{2}}(\alpha(x+n) - \alpha(x)) \right) + c.c. \right), \end{aligned} \quad (39)$$

where $\operatorname{erf}(z)$ is the error function. From here one derives uniform bounds both for $M_{XY}(h=0)$, for the coefficients A_k and for the correlation function [9]. Our result actually shows that such uniform bounds in L do not exist for the coefficients of the asymptotic expansion of the non-abelian model, if L is β -independent.

Appendix

The functions $G_{ll'}$ and $D_l(x')$ are defined as

$$G_{ll'} = 2\delta_{l,l'} - G_{x,x'} - G_{x+n,x'+n'} + G_{x,x'+n'} + G_{x+n,x'} , \quad (40)$$

$$D_l(x') = G_{x,x'} - G_{x+n,x'} = D(x - x' + n) - D(x - x') , \quad l = (x, n) , \quad (41)$$

where

$$D(x - y) = G_0 - G_{x,y} = \frac{1}{L^2} \sum_{k_n=0}^{L-1} \frac{1 - e^{\frac{2\pi i}{L} k_n (x_n - y_n)}}{2 - \sum_{n=1}^2 \cos \frac{2\pi}{L} k_n} , \quad k_n^2 \neq 0. \quad (42)$$

They satisfy the following equations

$$G_{l_1 l'} + G_{l_2 l'} - G_{l_3 l'} - G_{l_4 l'} = 0 , \quad (43)$$

$$D_{l_1}(x') + D_{l_2}(x') - D_{l_3}(x') - D_{l_4}(x') = 2\delta_{x,x'} , \quad (44)$$

$$\sum_x (G_{l_1 b_1} G_{l_2 b_2} + G_{l_3 b_1} G_{l_4 b_2} - G_{l_1 b_2} G_{l_2 b_1} - G_{l_3 b_2} G_{l_4 b_1}) = 0 , \quad (45)$$

where four links l_i are attached to a given site x . One proves the following “orthogonality” relations for the link functions

$$\sum_b G_{lb} G_{bl'} = 2G_{ll'} , \quad \sum_b D_b(x) G_{bl'} = 0 , \quad (46)$$

$$\sum_b D_b(x) D_b(x') = 2G_{x,x'} , \quad (47)$$

where the sum over b runs over all links of $2D$ lattice.

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